Feel like writing a review for The Mathematical Intelligencer? *Contact the column editor if you wish to submit an unsolicited review, or if you would welcome being assigned a book to review.*

➢ Submissions should be uploaded to http://tmin.edmgr.com
or to be sent directly to **Osmo Pekonen**,
osmo.pekonen@jyu.fi

A Brief History of Numbers

by Leo Corry

OXFORD: OXFORD UNIVERSITY PRESS, 2015, 336 PP., £24.99, ISBN: 9780198702597

REVIEWED BY JESPER LÜTZEN

The concept of number has an interesting and intriguing history and this is a highly recommendable book about it.

When were irrational numbers (or negative numbers) introduced into mathematics? Such questions seem precise and simple, but they cannot be answered with a simple year. In fact, the years 2000 BC and AD 1872 and many years in between could be a valid answer to the question about irrational numbers, and many years between 1 and 1870 could be defended as an answer to the question about the introduction of negative numbers. This is not just because modern mathematicians may consider pre-1870 notions of numbers as nonrigorous but also because the historical actors were not in agreement about the nature and extent of the concept of number. Therefore the answer to the questions raised previously must necessarily be a long and contorted story. Many treatises and articles on the history of mathematics tell aspects of this story, and Corry's book may be considered a synthesis of these more specialized works.

Still, Corry also adds his own original viewpoints and results to the standard history. The last chapters of the book dealing with the modern foundations of the number system from the end of the 19th and the beginning of the 20th centuries obviously draw on Corry's own research on the establishment of modern structural thinking that has contributed seminally to our understanding of the development of mathematics in this period. Moreover, Corry has also shed new light on the gradual merger of quantity, ratio, and number through a study of the transmission and edition of Euclid in the Renaissance.

However, the book is not primarily a research monograph but rather a semipopular book addressed to an audience of college students and mathematics teachers as well as high-school students and professional mathematicians. It is written in a pleasant, captivating style characteristic of the best popular mathematics books, but, because it is based on the most up-to-date scholarship, it paints an historically honest picture of the development of the concept of number. Many popular books on the history of mathematics try to make the presentation more readable by simplifying the historical account. Corry, on the contrary, does not cut corners. He presents the historical

© 2017 Springer Science+Business Media New York,
DOI 10.1007/s00283-017-9707-0

development in all its complexity, and that is exactly why the book is so captivating.

The main plot of the book is a story of how the ancient Greeks distinguished between numbers (collections of units: 2,3,4,...), magnitudes (for example, line segments), and ratios (between numbers or magnitudes) and how this distinction was afterward gradually blurred, leading to our modern concept of the real numbers. But this story is full of twists and turns and it has many side stories involving aspects of the history of algebra, geometry, and analysis. In addition to the real numbers, Corry also deals with the development of complex numbers, quaternions, and the infinite cardinals and ordinals. Infinitesimals, on the other hand, are only mentioned in passing. Number symbols and mathematical notation are discussed, but mostly only insofar as they have had an impact on the *concept* of number. Corry primarily deals with Western mathematics and the mathematics in cultures that have had an impact on Western mathematics, such as ancient Egypt and Babylon as well as Medieval Arab (Islamic) mathematics. Far Eastern and South American concepts of number are not discussed.

Because Corry wants to display the conceptual complexities in the history of numbers, he needs to handle a lot of technical mathematics. However, the technicalities are kept to a minimum and they are presented in a very pedagogical way, making them understandable to a good high-school student. Corry often engages in a dialog with his readers to urge them to delve into important mathematical arguments. For example, as an introduction to Cantor's diagonal argument for the uncountability of the real numbers, Corry writes: "I want to present Cantor's proof now. It is not particularly lengthy, and understanding its details does not require any specific mathematical background. If you find my explanation somewhat difficult to follow, I urge you to carry on bravely nonetheless to the end, if only for the sake of coming close to grasping the main ideas behind the famous argument." Some of the most technically demanding arguments have been relegated to appendices after each chapter. This holds, for example, for the proof of the irrationality of $\sqrt{2}$, Euclid's discussion of the area of a circle, Khayyam's geometric solution of the cubic equation, Dedekind's theory of cuts compared with Eudoxos's theory of proportions, and the proof that the set of algebraic numbers is countable.

When presenting old mathematical texts and ideas, an historian of mathematics must take into account the mathematical background of his/her modern reader. The historian may find it convenient to translate the old mathematics into modern terminology and symbolism to show the similarity of the old ideas with more familiar modern ideas. To a mathematician for whom sameness (isomorphism) is important, such translations are appealing. For an historian of mathematics, however, the development of mathematical ideas only shows itself when one focuses on the minute differences among different texts. Corry is very much aware of this problem. He often presents the old ideas in a modern translation or notation but he insists on the conceptual differences between this translation and the old original. In this way he shows subtle conceptual differences among different mathematical texts, and consequently he is able to tell a surprisingly sophisticated story. For example, I was struck by the minute differences Corry can discern in the development of the concept of a fraction.

In a book about the development of a mathematical concept, one has to distinguish between explicit definitions of mathematical objects and the more implicit conceptual ideas that one can extract from the technical use of the object. Corry repeatedly emphasizes the discrepancy between the two. For example, negative and complex numbers could be handled in a consistent manner long before the concepts had been properly defined or philosophically grasped. Similarly, fractions or irrational ratios were treated as numbers long before the definition of a number as a collection of units had been replaced by a new definition that would allow fractions to be considered as numbers. By insisting on contrasting the technical mathematical practice and the more foundational or philosophical aspects of a concept at a given time, Corry draws a richer and more complex picture than one customarily sees in popular books on the history of mathematics.

The book begins with a simple nonhistorical chapter on "The System of Numbers" that presents the various sets of numbers. It continues with a chapter on the idea of a positional system, illustrating the idea with the Egyptian, Babylonian, and Greek number signs.

Chapter 3 introduces the main dichotomy that must be kept in mind throughout the book, namely the Greek distinction between discrete numbers and continuous geometric magnitudes. The problem of incommensurability that necessitated this distinction is explained in a simple manner, and Corry explains how geometric magnitudes were not measured but compared in the Greek pure mathematical texts. He also explains the basic idea behind Eudoxos's theory of proportion (details in an appendix) and the Greek use of fractional numbers in more applied texts.

Chapter 4 deals with "Construction Problems and Numerical Problems in the Greek Mathematical Tradition." Euclid's number theory and Diophantus's numerical problems are discussed and are contrasted with Greek geometric problem solving. It is important to insist on this contrast because modern students will have encountered a different style of geometry in which the problems mostly concern finding the length of certain line segments or the measure of certain angles; that is, the problems ask for numbers and the solutions mostly involve calculation. As explained by Corry, classical Greek geometric problems ask for the construction (with ruler and compass or otherwise) of geometric objects. They involve no numbers or calculations. A modern reader will be tempted to translate many of the Greek geometric problems and their solutions into algebra (geometric algebra), but as pointed out by Corry one must keep in mind that such a translation only developed gradually during the subsequent two millennia.

The following chapter on the Medieval Islamic tradition does not handle the spread of the Hindu–Arabic decimal number system but mainly the emergence of algebraic techniques, the treatment of fractions, and Abu Kamil's and Al-Khayyam's uneasy attempts to merge numbers with geometric magnitudes. About Al-Khwarizmi's treatment of fractions, Corry concludes:

"All of this can be seen as direct evidence of how the arithmetic of fractions was practiced in the early stages of Islamic mathematics and how fractions were conceptualized. Ingenious ideas and innovative techniques taken from a variety of sources were tried out in original but hesitating ways, while their full integration into the broader body of existing arithmetic knowledge was yet to be formulated in a more coherent way that could also be combined with the classical conception of number as a collection of units."

Similar questions are pursued in the next chapter on "Numbers in Europe from the Twelfth to the Sixteenth Century." Corry discusses the development of mathematical (in particular algebraic) symbolism, the solution of the cubic equation, and the gradual acceptance of negative and complex numbers. Although these subjects are treated in many mathematics histories, Corry's analysis of Euclid's *Elements* in the Renaissance offers a novel and very enlightening perspective on the conception of number:

"One important element that gradually entered the editions of the *Elements*, as well as many other contemporary mathematical books, was the implicit assumption that any magnitude can be measured. Accordingly, the issue of incommensurability, which had been the main motivation for Eudoxus's formulation of a new theory of proportions… lost its importance.… As a consequence, there was a gradual identification of 'ratio' with the numerical value of the fraction defined by the two magnitudes compared."

This identification of magnitudes and numbers was made easier by Stevin's introduction of decimal fractions. They are discussed in the following chapter together with Viète's new general algebraic notation and the introduction of logarithms.

Chapter 8 is devoted to Descartes, Newton, and their contemporaries. Descartes's analytic geometry, in particular his arithmetic of line segments and his introduction of imaginary roots of equations, are discussed, as well as Wallis's preference for arithmetic instead of geometry and his strange graphical representation of complex numbers. The chapter ends with a very fine and many-faceted analysis of Newton's *Universal Arithmetick*. According to Newton, "by number we understand, not so much a multitude of Unities, as the abstracted ratio of any Quantity, to another quantity of the same Kind, which we take for unity." As pointed out by Corry, "The unit, the integers, the fractions, and the irrational numbers appear here—perhaps for the first time and certainly in an influential text in such clear-cut terms—all as mathematical entities of one and the same kind… Moreover, and very importantly, numbers are abstract entities; themselves they are not quantities, but they may represent either a quantity or a ratio between quantities."

In the 18th century, metaphysical considerations about the nature of numbers were generally giving way to the question of how numbers work. This is very visible in the works of Euler and d'Alembert. And yet it was at the end of that century and at the beginning of the 19th century that the geometric interpretation offered some kind of understanding of what complex numbers could be. This story and Hamilton's invention of quaternions are the main subjects discussed in Chapter 9.

In Chapter 10, Corry has reached the late 19th century and thus the period to which his own research has contributed in a major way. The chapter deals primarily with Dedekind's use of infinite sets, fields, and ideals to build up the rational numbers, and his use of cuts to construct the real numbers, overcoming thus for the first time the divide between the discrete numbers and the continuous magnitudes (and even giving a satisfactory characterization of continuity or completeness).

As a conclusion of the main development of the concept of number, Chapter 11 reports on the axiomatic foundation of the natural numbers. Here the ideas of Dedekind, Peano, and Frege are discussed and compared. A chapter follows on Cantor's theory of transfinite ordinals and cardinals as well as the antinomies resulting from the naïve point of view and the axiomatic attempts to fix the problems.

In a short epilogue, Corry emphasizes the interplay between the driving forces of mathematics (problems, be they internal to mathematics or external) and the foundations of mathematics including the concept of number. In the course of the book he often calls attention to cases in which mathematical practice and mathematical problems have necessitated subsequent conceptual changes. He points out that this dynamic is in agreement with a quote by Hilbert concerning the healthy path of mathematical development that Corry has himself made famous.

As is commonplace in popular books, there are only a few references in the main text. Instead there is a useful section with suggestions for further reading, beginning with general works and continuing with works related to specific chapters. This list contains some older classics but most of the references are recent works, including both more general treatises or source books and specialized papers.

There are a few irritating misprints and other mistakes in the book. For example, on page 34 oblong numbers are defined as numbers of the form $n - (n + 1)$ (rather than $n\,(n + 1)$). On page 147, Bombelli's rule "meno di meno via più di meno fa più" is translated into $\sqrt{-1} \times \sqrt{-1} = 1$. It should have been $(-\sqrt{-1}) \times \sqrt{-1} = 1$. On page 157, Corry accidentally wrote that Viète used consonants for the unknowns and vowels for the known quantities. As becomes clear in the examples that follow, it is the reverse. On page 160, there is a wrong sign, and on page 189 the square should have side 1 not 2. Finally, some text at the end of section 11.3 seems to be missing.

Of course a book of a modest size on a vast subject will always have to skip some subjects. I would have liked to hear more in two places. One is the interesting consideration in the so-called Merton school concerning ratios and fractions. The other is a more refined discussion of complex and imaginary numbers in the work of Descartes and his followers. Indeed, Corry leaves the impression that imaginary numbers are the same as complex numbers (that is, numbers of the form $a + b\sqrt{-1}$). However, it was only with Euler and d'Alembert that mathematicians generally agreed that all the imagined roots of an equation must have that form. This adds an interesting complexity to the story of complex numbers.

I conclude this review by warmly recommending the book to any mathematically interested person who wants to know how the concept of number has emerged and who is not content with a digest version. In fact, the intricate complex version is much more interesting, at least when it is written by Leo Corry.

Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København
Denmark
e-mail: lutzen@math.ku.dk